# Solving Portfolio Optimization Problems Using MOEA/D and Lévy Flight

Knowledge System Lab

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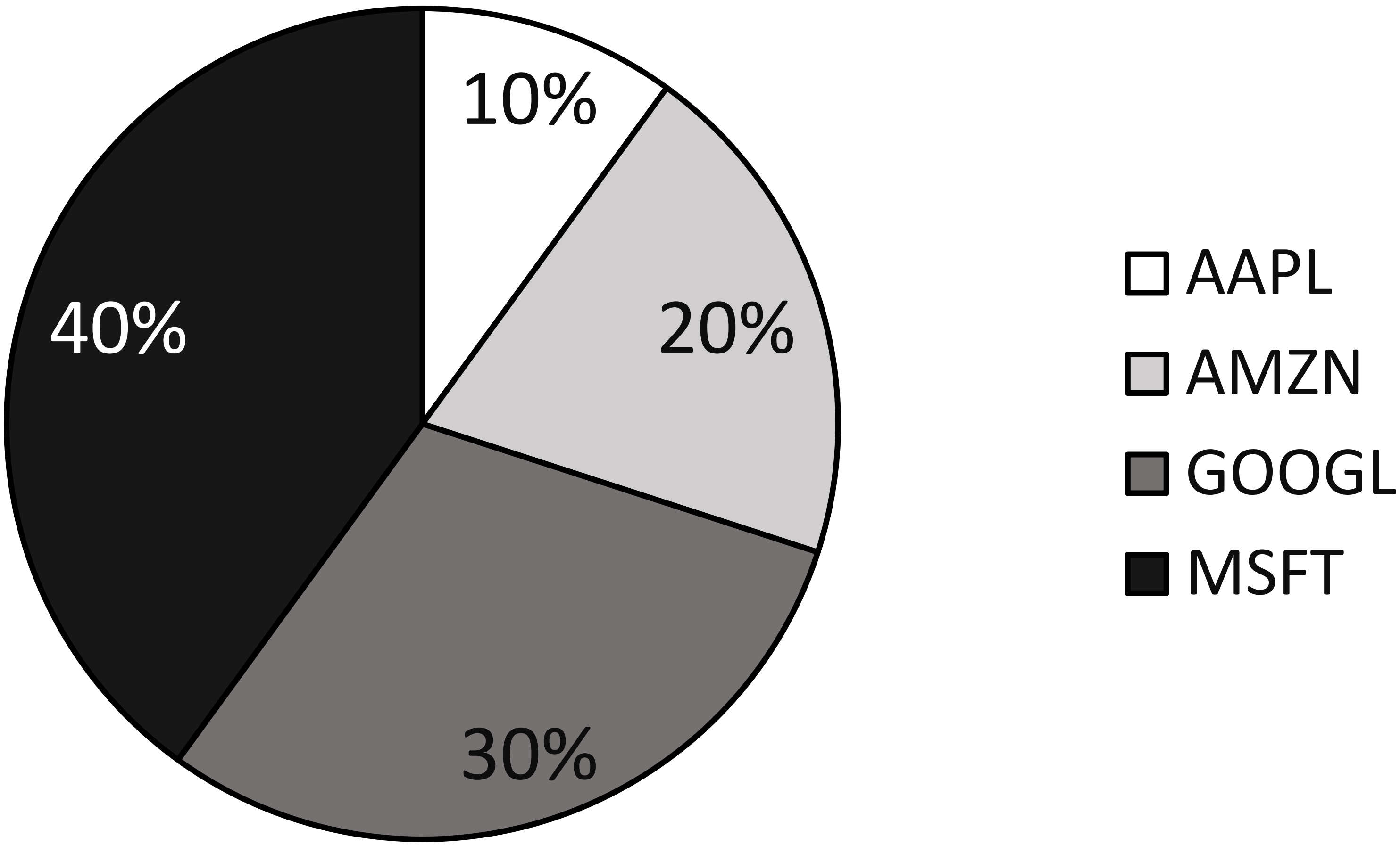
# Outline

*Lévy Flight keeps a good balance between local and global search. Furthermore, it utilizes structural information of the problem when solving portfolio optimization.*

- Background
- Related Work
- Proposed Method
- Experiments

# An example of portfolio on four stocks



# Description of Portfolio

- One single stock

Get many money
Get little money
Lose little money
Lose many money

} Return as a random variable $R_i$

- With normal assumption,

$E(R_i)$ — Average return

$V(R_i)$ — Chance of a low return = Risk

# Description of Portfolio

Mean-Variance Portfolio Model [Markowitz 1954]

- Portfolio: weighted sum of stocks

$$R = \sum_{i=1}^{n} w_i \cdot R_i$$

$w_i$: $i$-th invest weight   $R_i$:$i$-th asset return

- Portfolio return: expectation value of a sum

$$E(R) = \sum_{i=1}^{n} w_i \cdot E(R_i)$$

- Portfolio risk: variance of a sum

$$V(R) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot Cov(R_i, R_j)$$

# Unconstrained PO Problem

- Given $n$ stocks holding returns $\{R_1, \cdots, R_n\}$
- Find a portfolio weight $\{w_1, \cdots, w_n\}$
  - Maximize portfolio return

$$\sum_{i=1}^{n} w_i \cdot E(R_i)$$

  - Minimize portfolio risk

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot Cov\left(R_i, R_j\right)$$

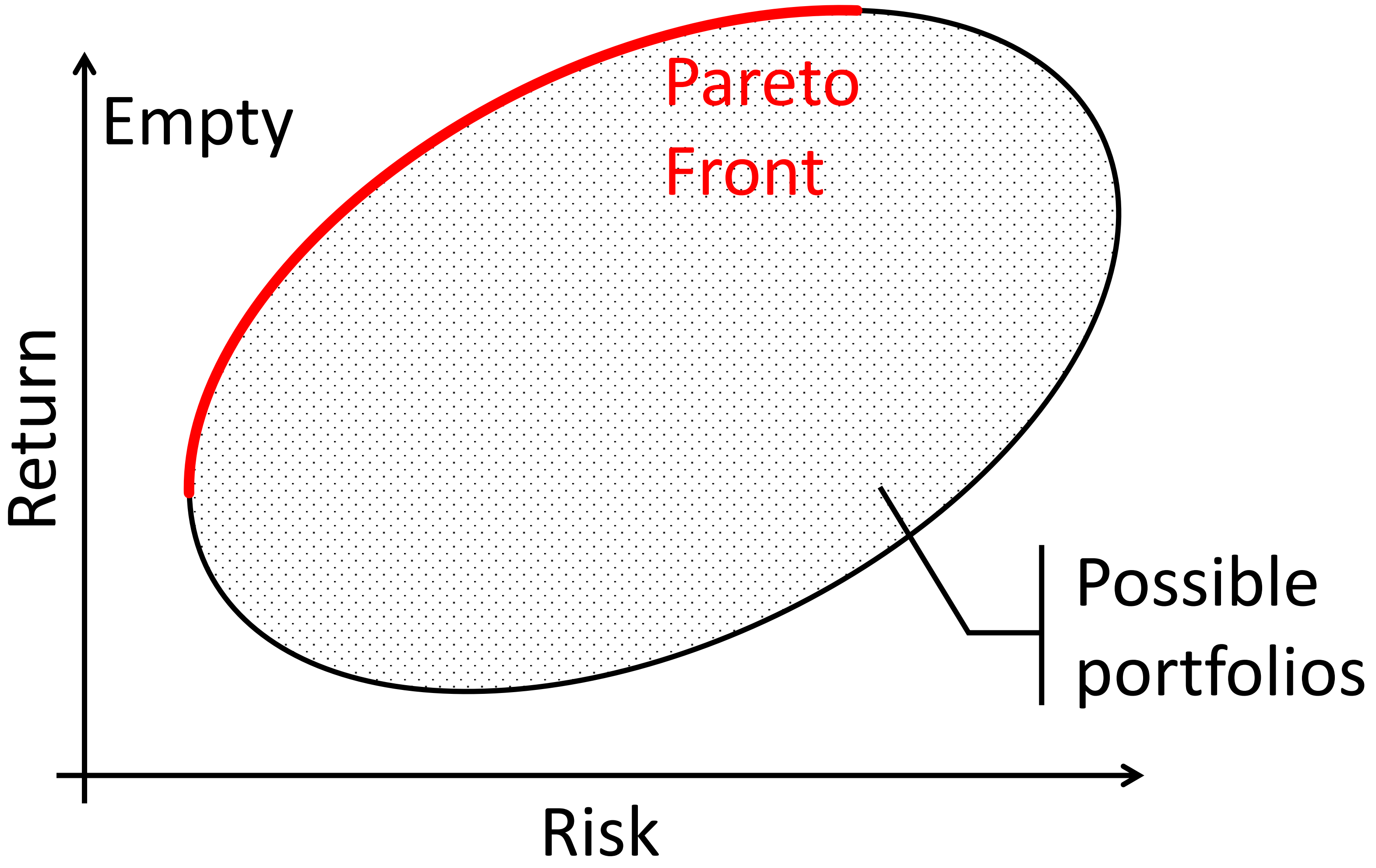
- Subject to,
  - Unit constraint $\sum_{i=1}^{n} w_i = 1$
  - No short selling $w_i \geq 0$

# Importance of PO Problem

- Practical meaning (it makes money)

- Theoretical meaning (it is hard)
  - High dimension
  - NP-hard with proper constraints [Bienstock 1996]

# Optimal Portfolios




Multi-Objective Optimization Problem (MOOP)

# MOEA based on Decomposition (MOEA/D) [Zhang 2007]

- E.g. PO: $\{\max E, \min V\} \Rightarrow \min\{-E, V\}$
- $s_1 = -0.0E + 1.0V$
- $s_2 = -0.2E + 0.8V$
- $s_3 = -0.4E + 0.6V$
- $s_4 = -0.6E + 0.4V$
- $s_5 = -0.8E + 0.2V$
- $s_6 = -1.0E + 0.0V$

Sub-problems

$\boldsymbol{x}^{(\boldsymbol{i})}$: solution of $s_i$

Borrow from neighbors

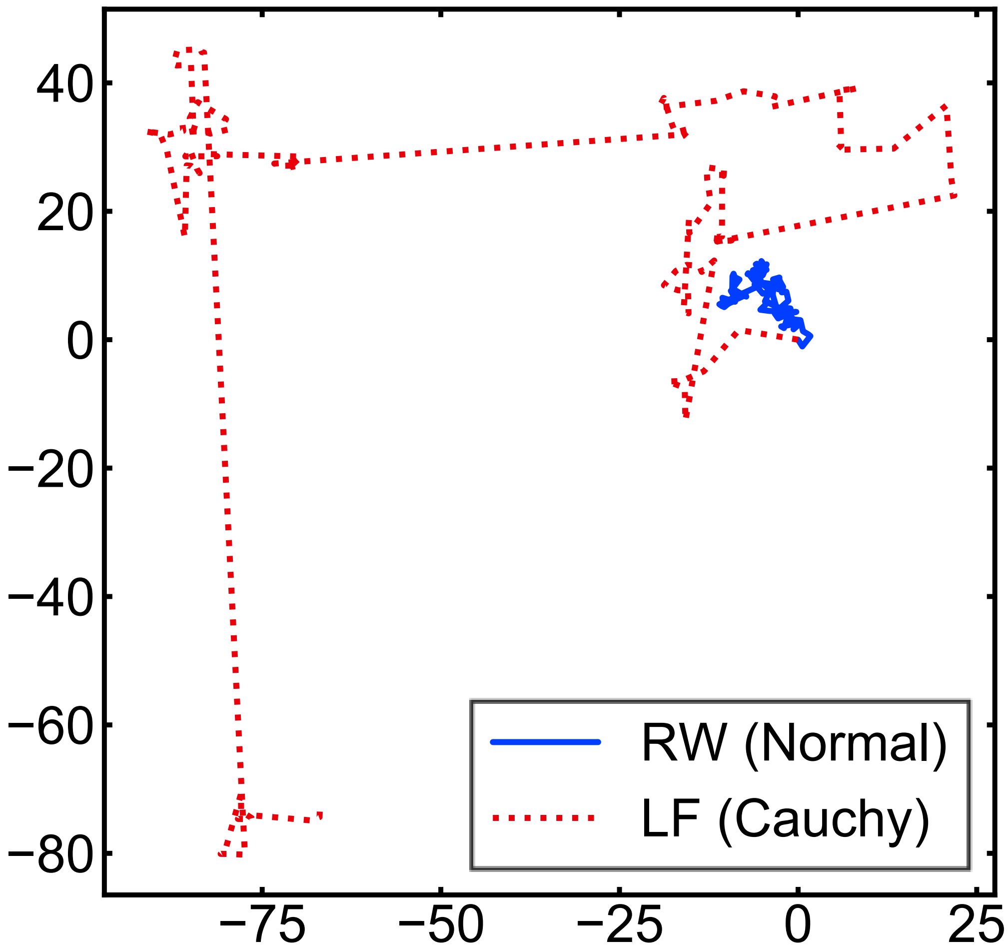
- Before reproduction: Select from neighbors
- After reproduction: Update neighbors by offspring

# Mutation Methods in MOEA/D

| | Algorithm | Mutation | Publication |
|---|---|---|---|
| MOEA/D- | PSO | Particle Swarm Optimization | [Peng 2008] |
| | DE | Differential Evolution | [Li 2009] |
| | ACO | Ant Colony Optimization | [Ke 2013] |

- MOEA/D-DE [Li 2009]
  - DE mutation: $\boldsymbol{y} = \boldsymbol{x}^{(\boldsymbol{i})} + F \cdot \left(\boldsymbol{x}^{(\boldsymbol{j})} - \boldsymbol{x}^{(\boldsymbol{k})}\right)$
  - Polynomial mutation

# Lévy Flight (LF): A Comparison with Random Walk




- Random walk (RW) whose step length follows stable distribution (heavy-tailed)

100 Steps of RW and LF from (0,0) in 2D Space

# Related Work: Solving PO with MOEAs

- Non-MOEA/D methods
  - SPEA2, NSGA-II [Mishra 2011]
  - NS-MOPSO [Mishra 2014], MOBFO [Mishra 2014], M-CABC [Kumar 2017]
- MOEA/D-based methods
  - Mutation: MOEA/D-DE [Zhang 2010] (with DE mutation only in reproduction)
  - Weight vector: MOEA/D-CP [Zhang 2018]
  - Initialization: DEA-MOEA/D [Zhou 2019]

# Proposed Method: MOEA/D-Lévy

- Inject LF mutation into MOEA/D:
  - $\boldsymbol{y} = \boldsymbol{x}^{(\boldsymbol{i})} + \alpha_0 \cdot \left(\boldsymbol{x}^{(\boldsymbol{i})} - \boldsymbol{x}^{(\boldsymbol{j})}\right) \oplus \boldsymbol{Levy}(\boldsymbol{\beta})$

    $\oplus$ : entry-wise multiplication

    $\alpha_0$: scaling factor

    $\beta \in [0.3, 1.99]$: shape parameter
  - Polynomial mutation
- Repair steps
  - Set negative variables to 0
  - Scale to satisfy unit constraint

# Experimental Settings

- Data: PO benchmark in OR library (Nikkei, size=225) [Chang 2000]

- Parameters, convergence settings are fine-tuned or set according to prior studies

- Evaluation Metrics (totally six)
  - Inverted Generation Distance (IGD)
  - Hypervolume (HV)

# Comparison with Literature Methods

| Method | | Mutation | Publication |
|---|---|---|---|
| MOEA/D- | Lévy | LF mutation+ Polynomial mutation | Proposed method |
| MOEA/D- | DEM | DE mutation + Polynomial mutation | [Li 2009] |
| MOEA/D- | DE | DE mutation only | [Zhang 2010] |
| MOEA/D- | GA | GA reproduction | [Zhang 2007] |
| NSGA-II | | GA reproduction | [Deb 2002] |

# Experimental Results on Nikkei

| Metric | MOEA/D -Lévy | MOEA/D -DEM | MOEA/D -DE | MOEA/D -GA | NSGA-II |
|---|---|---|---|---|---|
| IGD | | | | | |
| Best | 1.77e-05 | 1.90e-05 | 7.90e-05 | 1.77e-04 | 4.64e-05 |
| Median | **2.39e-05** | 2.73e-05 | 2.23e-04 | 2.41e-04 | 9.69e-05 |
| Std. | 1.15e-05 | 4.09e-04 | 6.95e-05 | 5.29e-04 | 3.67e-05 |
| HV | | | | | |
| Best | 8.31e-06 | 8.29e-06 | 7.96e-06 | 7.87e-06 | 8.19e-06 |
| Median | **8.29e-06** | 8.26e-06 | 7.23e-06 | 7.54e-06 | 7.94e-06 |
| Std. | 1.59e-08 | 9.52e-07 | 3.43e-07 | 1.20e-06 | 1.08e-07 |

*The special font shows **Best** and **Statistical significance**

# Comparison with Other Distribution-based Mutation

| Method | Mutation |
|---|---|
| LEVY | $\boldsymbol{y} = \boldsymbol{x}^{(i)} + \alpha_0 \cdot \left(\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(j)}\right) \oplus \boldsymbol{Levy(\beta)}$ |
| UNIF | $\boldsymbol{y} = \boldsymbol{x}^{(i)} + C \cdot \left(\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(k)}\right) \oplus \boldsymbol{Unif(-1,1)}$ |
| NORM | $\boldsymbol{y} = \boldsymbol{x}^{(i)} + C \cdot \left(\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(k)}\right) \oplus \boldsymbol{N(0,1)}$ |
| CONST | $\boldsymbol{y} = \boldsymbol{x}^{(i)} + F \cdot \left(\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(k)}\right) \oplus \boldsymbol{1}$ |

# Tracking Successful Trajectories

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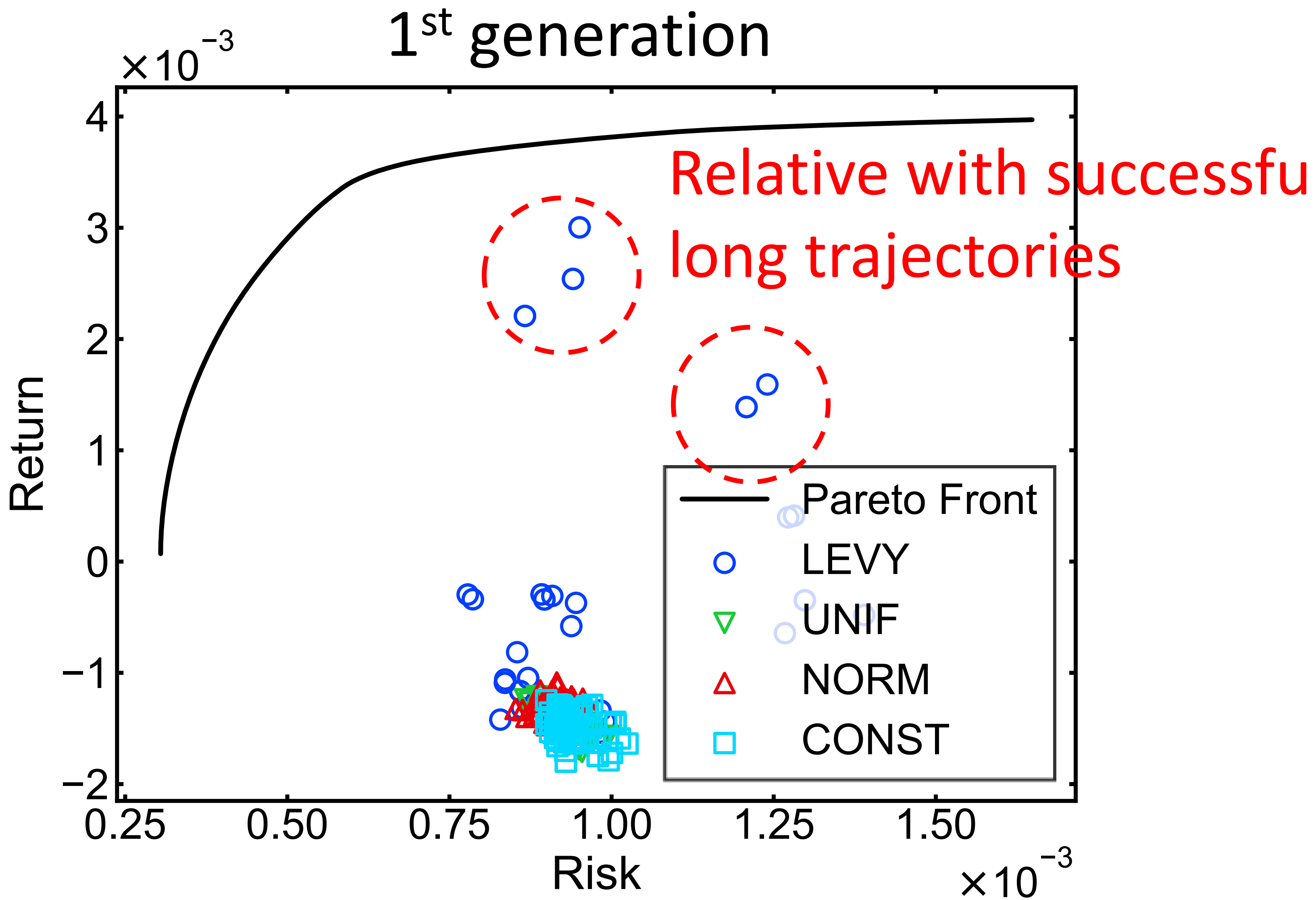

# Trajectories as Capital Re-allocation in PO

| Strategy | Asset Type | Portfolio |
|---|---|---|
| Random | - | Low risk |
| Centralized | High return | High return |

Heavy-tailed

- Many small values
- Few large values

Re-allocation by LF is centralized!

# Trajectories as Capital Re-allocation in PO

Parent

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.10 | 0.15 | 0.20 | 0.15 | 0.18 | 0.02 | 0.20 |

Mutation (Re-allocation) Vector by LF

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.05 | **5.00** | 0.10 | 0.50 | 0.08 | 0.02 | 0.67 |

Offspring before Repair

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.15 | **5.15** | 0.30 | 0.65 | 0.26 | 0.04 | 0.87 |

Offspring after Repair

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.02 | **0.69** | 0.04 | 0.09 | 0.03 | 0.01 | 0.12 |

# Conclusions

- Summary of the Research
  - Inject LF into MOEA/D and assess on PO
  - Relation between long trajectories and moving solution to high return areas
  - Explain by considering property of PO
- Future Work
  - Adaptive strategy
  - Solve sub-problems with different mutation methods